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STRESS INTENSITY FACTORS AND CRACK DEVIATION CONDITIONS IN A BRITTLE ANISOTROPIC SOLID

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For arbitrary anisotropy in the linear manifold of singular solutions generating square-root singularities of the crack tip stress, a special basis is introduced that possesses the same properties as in the isotropic case and provides simple integral representations for the attributes of the energy fracture criterion, in particular, the conditions of crack deviation from a straight path.

Key words: *crack, intensity factors, brittle anisotropic solid.*

1. Normalizations of Singular Solutions. For isotropic and some orthotropic planar elastic media, direct calculations (see [1–3] and other papers) show that the stresses at the crack tip $\mathcal{O}$ have square-root singularities $O(r^{-1/2})$. In this case, the angular parts of the corresponding displacement vectors

$$U^j(x) = r^{1/2}\Phi^j(\varphi) \qquad (j = 1, 2) \tag{1.1}$$

are chosen according to the classical definition of the stress intensity factors (SIF)

$$K_i = \lim_{x\to+0}(2\pi r)^{1/2}\sigma_{3-i,2}(u; x_1, 0) \qquad (i = 1, 2). \tag{1.2}$$

In (1.1) and (1.2), $x = (x_1, x_2)$ and (r, φ) are Cartesian and polar coordinates with origin $\mathcal{O}$; the positive semiaxis $\mathcal{O}x_1$ and the polar axis are at the crack continuation, along which the limit (1.2) is calculated; $\sigma_{jk}(u)$ are the Cartesian components of the stress tensor found from the displacement field $u = (u_1, u_2)$. In [4, 5] it is established that the square-root singularity is retained for any anisotropy, and in [6], it is verified that the definition of the SIF (1.2) remains consistent. The latter verification is fairly simple. If for the displacements (1.1), the stresses $\sigma_{k2}(U^j; x_1, 0)$ depend linearly on $\mathcal{O}x_1$, there exists a field $U(x) = r^{1/2}\Phi(\varphi)$ such that

$$\sigma_{21}(U; x_1, 0) = \sigma_{22}(U; x_1, 0) = 0, \qquad x_1 > 0. \tag{1.3}$$

Together with the equilibrium equations and boundary conditions

$$\begin{gathered} -\frac{\partial}{\partial x_1}\,\sigma_{1k}(U; x) - \frac{\partial}{\partial x_2}\,\sigma_{2k}(U; x) = 0, \qquad x \in \mathbb{R}^2 \setminus \Lambda, \quad k = 1, 2, \\ \sigma_{21}(U; x_1, 0) = 0, \qquad \sigma_{22}(U; x_1, 0) = 0, \qquad x_1 < 0, \end{gathered} \tag{1.4}$$

where $\Lambda = \{x \in \mathbb{R}^2\colon\ x_1 \le 0,\ x_2 = 0\}$ is a semi-infinite cut, relations (1.3) show that U is a solution of the homogeneous elasticity problem in the upper half-plane that is bounded in any neighborhood of the point $\mathcal{O}$. As is known, this solution is smooth, i.e., the singularity $O(r^{1/2})$ is forbidden. The above contradiction shows that one can always match the basis (1.1) of singular solutions to the normalization conditions

$$\sigma_{3-i,2}(U^j; r, 0) = \delta_{i,j}(2\pi r)^{-1/2}, \qquad i, j = 1, 2 \tag{1.5}$$

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and thus obtain a correct definition of the two SIFs — K_1 and K_2. On the right side of (1.5), $\delta_{i,j}$ is the Kronecker symbol.

The basis (1.1) which satisfies conditions (1.5) should be considered adapted to the force fracture criteria proposed by Irwin, Novozhilov, and others. In [6, 7], we proposed bases adapted to the energy and deformation criteria, respectively. In the second case, the normalization condition that replace (1.5) are written as

$$(1/2)[U_i^j](-r) = 4(2\pi)^{-1/2} B_{11,11}\delta_{3-i,j} r^{1/2}, \qquad i,j = 1,2. \tag{1.6}$$

Here $[u](x_1) = u(x_1, +0) - u(x_1, -0)$ is the jump of the displacement vector on the crack faces Λ; $B_{jk,pq}$ are the elements of the pliability tensor, which is the reverse of the stiffness tensor A in Hooke's law $\sigma = A\varepsilon$ (ε is the strain tensor). As before, the possibility of satisfying conditions (1.6) is established by contradiction: if the displacement field $U(x) = r^{1/2}\Phi(\varphi)$ has a zero jump at the cut Λ, then, by virtue of (1.4), it is a solution of the equilibrium equations on the punctured plane $\mathbb{R}^2 \setminus \mathcal{O}$, and, hence, can have only an integer homogeneity factor but not 1/2. The reasons for the occurrence of the element $B_{11,11}$ on the right side of (1.6) instead of the more natural elements $B_{22,22}$ and $B_{12,12}$ will become clear below.

The multipliers on the right side of (1.6) (they were not introduced in [7]) are chosen so that for an isotropic solid, the normalizations (1.5) and (1.6) give the same basis (1.1). In the anisotropic case, the bases $\{U^{j\sigma}\}$ and $\{U^{j\varepsilon}\}$, generally speaking, differ. For them and for the factors K_j^σ and K_j^ε in the expansions near the crack tip $\mathcal{O}$, the displacement fields

$$u(x) = K_1^\sigma U^{j\sigma}(x) + K_2^\sigma U^{j\sigma}(x) + \ldots = K_1^\varepsilon U^{1\varepsilon}(x) + K_2^\varepsilon U^{2\varepsilon}(x) + \ldots \tag{1.7}$$

have constraints

$$U^{j\sigma} = T_{1j}U^{1\varepsilon} + T_{2j}U^{2\varepsilon}, \qquad K_j^\varepsilon = K_1^\sigma T_{j1} + K_2^\sigma T_{j2}. \tag{1.8}$$

The elements T_{jk} of the 2×2 matrix T depend on both the elastic properties of the material and the crack direction [see (4.7) below]. We note that according to the requirement (1.6) the factors (1.7) in the second representation are found from the formula

$$K_i^\varepsilon = (1/8)(2\pi)^{1/2} B_{11,11}^{-1} \lim_{x\to -0} r^{-1/2}[u_{3-i}](x_1), \qquad i = 1,2. \tag{1.9}$$

According to [8] (see also [6, Sec. 4]), for each basis (1.1) one can uniquely define the basis of the weight functions

$$V^k(x) = r^{-1/2}\Psi^k(\varphi) \qquad (k = 1,2), \tag{1.10}$$

for which the following biorthogonality conditions are satisfied:

$$Q(U^j, V^k; \Gamma) = \delta_{j,k}, \qquad j,k = 1,2. \tag{1.11}$$

Here Q is an antisymmetric [i.e., $Q(u,v;\Gamma) = -Q(v,u;\Gamma)$] form that arises as the contour integral in the Green's formula

$$Q(u,v;\Gamma) = \int_\Gamma \{v(x)\cdot\sigma^{(n)}(u;x) - u(x)\cdot\sigma^{(n)}(v;x)\}\,ds, \tag{1.12}$$

where ds is the element of the length of the simple arc Γ that connects the crack faces $\Lambda^\pm$ of the cut and encompasses the tip $\mathcal{O}$; $\sigma^{(n)} = \sigma n$ is the normal stress vector; n is the unit outward normal to the boundary of the domain contained inside Γ. If u and v satisfy problem (1.4), then, by virtue of Green's formula, integral (1.12) does not depend on the path Γ. In addition, in [9] (see also [6, Sec. 4]), we proved a formula that resembles the rule of integration by parts, namely,

$$Q(\partial_1 u, v; \Gamma) = -Q(u, \partial_1 v; \Gamma), \tag{1.13}$$

where $\partial_1 = \partial/\partial x_1$ is the differentiation along the crack.

The derivative $\partial_1 U^j$ still solves the problem (1.4) but acquires singularities $O(r^{-1/2})$ at the crack tip and, consequently,

$$\partial_1 U^j(x) = -M_{j1}V^1(x) - M_{j2}V^2(x), \qquad j = 1,2. \tag{1.14}$$

By virtue of relations (1.11) and (1.13), the following integral representations are valid:

$$M_{jk} = Q(\partial_1 U^j, U^k; \Gamma) = Q(\partial_1 U^k, U^j; \Gamma) = M_{kj}, \tag{1.15}$$

i.e., the 2×2 M matrix composed of the coefficients in expansion (1.14) is symmetric. In addition, its positive definiteness was established in [9, 6]. Hence, one can find a basis $\{U^{je}\}$ that satisfies the equalities $\partial_1 U^{je} = -m_j V^{je}$, where $m_j > 0$ are positive eigenvalues of the matrix M. This basis is related to the energy fracture criterion (see [6, Sec. 5]) and has representations similar to (1.7) and (1.8).

2. Additional Properties of the Strain Basis. Everywhere below, we use the notation $U^j = U^{j\varepsilon}$. If $\{x : x_2 = 0\}$ is a plane of elastic symmetry, then U^j_k are even function of the variable for $j \neq k$ and even function for $j = k$. In particular,

$$U^1_1(x_1, \pm 0) = 0, \qquad U^2_1(x_1, +0) = -U^2_1(x_1, -0) \qquad \text{at} \quad x_1 < 0. \tag{2.1}$$

One more observation refers to the isotropic case: explicit formulas for the first mode U^1 (see, for example, [2, p. 316]) show that

$$\sigma_{11}(U^1; x_1 \pm 0) = 0 \qquad \text{at} \quad x_1 < 0. \tag{2.2}$$

In other words, according to boundary conditions (1.4) for $x_1 < 0$ and formula (2.2), all stresses on the crack faces $\Lambda^\pm$ are eliminated, and, hence, because of the equilibrium equations, the following formulas are valid:

$$\begin{gathered} \partial_2 \sigma_{k2}(U^1; x_1, \pm 0) = -\partial_1 \sigma_{k1}(U^1; x_1, \pm 0) = 0, \qquad x_1 < 0, \quad k = 1, 2, \\ \partial_2^2 \sigma_{22}(U^1; x_1, \pm 0) = -\partial_1 \partial_2 \sigma_{k1}(U^1; x_1, \pm 0) = \partial_1^2 \sigma_{11}(U^1; x_1, \pm 0) = 0, \qquad x_1 < 0. \end{gathered} \tag{2.3}$$

The immediate objective is to prove relations (2.1)–(2.3) for arbitrary anisotropy.

Let e be a tensor with Cartesian components $e_{11} = 1$ and $e_{pq} = 0$ for $p + q > 2$. Since $B_{11,11} = e \cdot Be$ and $\sigma(U^j) = e\sigma_{11}(U^j)$ on $\Lambda^\pm$ by virtue of boundary conditions (1.4), we find that

$$\begin{aligned} \partial_1 U^j(x_1, \pm 0) &= \varepsilon_{11}(U^j; x_1, \pm 0) = e \cdot \varepsilon(U^j; x_1, \pm 0) \\ &= e \cdot B\sigma(U^j; x_1, \pm 0) = B_{11,11}\sigma_{11}(U^j; x_1, \pm 0), \qquad x_1 < 0. \end{aligned} \tag{2.4}$$

Thus, equality (2.2) and then equalities (2.3) are derived from the properties of the basis (2.1). To verify these properties, we substitute the fields $\partial_2 U^2$ and U^j into Green's formula for an open ring $\Xi = \{x\colon\ 0 < a < r < b,\ \varphi \in (-\pi, \pi)\}$. Using Eq. (1.4), we obtain

$$Q(\partial_2 U^2, U^j; \Gamma_b) - Q(\partial_2 U^2, U^j; \Gamma_a) = \sum_\pm \pm \int_a^b \sum_{k=1}^2 U^j_k(-r, \pm 0)\sigma_{k2}(\partial_2 U^2; -r, \pm 0)\, dr. \tag{2.5}$$

In this case, Γ_ρ is a circle of radius ρ and since for $u = U^j$ and $v = \partial_2 U^2$, the integrand in (1.12) is $O(r^{-1})$, the integral $Q(\partial_2 U^2, U^j; \Gamma_\rho)$ does not depend on ρ, i.e., the left side of (2.5) is equal to zero. In addition, $\partial_2 \sigma_{k2}(U^2) = -\partial_1 \sigma_{k1}(U^2)$ and, hence, using boundary conditions (1.4) for $x_1 < 0$ and equality (2.4), we bring formula (2.5) to the form

$$0 = B^{-1}_{11,11} \sum_\pm \pm \int_a^b U^j_1(-r, \pm 0)\partial_1^2 U^2_1(-r, \pm 0)\, dr = B^{-1}_{11,11} \ln \frac{a}{b} \{\Phi^j_1(+\pi)\Phi^2_1(+\pi) - \Phi^j_1(-\pi)\Phi^2_1(-\pi)\}.$$

The vanishing of the difference in braces implies that

$$[U^j_1](-r) \sum_\pm U^2_1(-r, \pm 0) + [U^2_1](-r) \sum_\pm U^j_1(-r, \pm 0) = 0,$$

i.e., the sought relations (2.1) are satisfied by virtue of the normalization (1.6).

We note an interesting consequence of formulas (2.3), by virtue of which the derivative $\partial_2 U^1$ (the differentiation is performed across the cracks!) is a power-law solution of the problem (1.4) but acquires a singularity $O(r^{-1/2})$ at the tip $\mathcal{O}$. Since, according to (2.2), the strain tensor $\varepsilon(U^1)$ is eliminated on the crack faces, taking into account conditions (1.6), we define the jumps

$$[\partial_2 U^1_1](-r) = 2[\varepsilon_{12}(U^1)](-r) - [\partial_1 U^1_2](-r) = 4(2\pi)^{-1/2} B_{11,11} r^{-1/2},$$

$$[\partial_2 U_2^1](-r) = [\varepsilon_{22}(U^1)](-r) = 0.$$

They are the same as those for $-\partial_1 U^2$; hence, because the power-law solutions with an exponent of $-1/2$ [weight functions; see (1.10) and (1.14)] are uniquely expanded over the basis $\{\partial_1 U^j\}$, the following equality holds:

$$\partial_2 U^1 = -\partial_1 U^2. \tag{2.6}$$

3. Determining the Stress Intensity Factors. In [10–12, 7] and other papers, relations (2.1), (2.2), and (2.6), found for an isotropic material by direct calculations, were used to form and employ the weight functions and invariant integrals, including those of higher orders, to determine the distortion of crack paths by (small) shear loads. It is remarkable that two constraints that were not noted in (2.1) but are provided by the symmetry properties in the isotropic case were not used the papers cited. This forces us to adopt the strain basis of singular solutions subject to the normalization conditions (1.6) as the principal basis.

To impart the meaning of SIF to the factors K_j^ε in formulas (1.7) and (1.9), we write relations (1.6) using only stresses. From (2.4) and (1.6) it follows that

$$[\sigma_{11}(U^j)](-r) = -4(2\pi r)^{-1/2}\delta_{2,j}, \qquad j = 1, 2. \tag{3.1}$$

In view of the identity (2.2), U^1 cannot be uniquely found from the above formula, but owing to relations (2.6) and (3.1), we obtain one more formula

$$[\partial_2\sigma_{11}(U^1)](-r) = -[\partial_1\sigma_{11}(U^2)](-r) = 2(2\pi)^{-1/2}r^{-3/2}. \tag{3.2}$$

We emphasize that $\partial/\partial x_1 = -\partial/\partial r$ on the cut Λ. The basis $\{U^{j\varepsilon}\}$ and the SIFs K_j^ε are now determined. We note that according to the second equality in (2.1), the left side of (1.6) for $j = 2$ can be replaced by $\pm U_1^2(x_1, \pm 0)$ and for the normalization of U^1, we use formula (2.6) instead of (3.2).

The above calculations and conclusions remain valid for a crack on an interface between anisotropic media provided that the stress singularity exponents remain real (see [6] and the references therein). By virtue of the requirement (1.6), the relation $K_1 > 0$ provides crack mouth opening (no contact of the faces near the tip $\mathcal{O}$) and satisfaction of unilateral constraints in the Singorini problem:

$$\begin{gathered}\sigma_{21}(U; x_1, \pm 0) = 0, \qquad [\sigma_{22}(U)](x_1) = 0, \qquad \sigma_{22}(U)(x_1, \pm 0) \leqslant 0, \\ [U_2](x_1) \geqslant 0, \qquad [U_2](x_1)\sigma_{22}(U)(x_1, \pm 0) = 0, \qquad x_1 < 0.\end{gathered} \tag{3.3}$$

We emphasize that in the presence of power-law solutions $r^{\pm i\gamma + 1/2}\Phi^\pm(\varphi)$ with $\gamma > 0$, relations (3.3) are violated for any nonzero (complex) SIFs, i.e., one cannot do without a complete solution of the Singorini problem — in fact in [13, 14], a nonlinear problem is solved although linear constitutive relations are used. In this connection, we note papers [15, 16], in which the rate of liberation of potential strain energy is found for straight crack growth with possible contact between its faces.

One more advantage of the strain basis $\{U^{j\varepsilon}\}$ over the force basis $\{U^{j\sigma}\}$ is the preservation of $U^{2\varepsilon}$ as a (unique) power-law solution in the model problem of a crack with faces in contact (cf. [13, 14]; see also boundary conditions (3.3), where the "$\geq$" and "$\leq$" signs are replaced by "="). In addition, the element $U^{2\varepsilon}$ does not need to be rearranged for transformations that implement the algebraic equivalence of anisotropic media [17, 3].

4. Crack Deviation Condition. According to the general results of [18; 19, Chapter 7], which are adapted in [6, 7] to linear elasticity problems, the rate of energy liberation due to the formation of a branch of small length $h > 0$ directed at an angle θ from the tip of a main crack is found from the formula

$$\Delta U(h, \theta) = -\frac{1}{2}\, h \sum_{j,k=1}^{2} \mathcal{M}_{jk}(\theta) K_j K_k + O(h^{3/2}) \qquad (h \to +0), \tag{4.1}$$

in which $\Delta U(h, \theta)$ is the increment in the potential strain energy and $K_j = K_j^\varepsilon$ is the SIF (1.9) in formula (1.7) for solving the problem of a crack in the initial position. In [6], it is shown that

$$\mathcal{M}(0) = M, \tag{4.2}$$

and the elements of the matrix M are taken from relations (1.14) and (1.15). The same paper gives another representation for $\Delta U(h, \theta)$, which is interpreted as *a posteriori* Griffith's formula and is related to one of the invariant integrals.

The relationship between the matrices M and T is found from formulas (1.14), (4.2) and (1.7), (1.8) using the asymptotic representation (4.1). We denote by u and u^h the solutions of the problem of deformation of a solid Ω with boundary straight cracks Λ and Λ^h in the absence of body forces under identical surface loads g applied to the outer boundary $\partial\Omega$ and equal to zero on the crack faces. Substituting u and u^h into Green's formula for the domain $\Omega \setminus \Lambda^h$, we obtain the equality

$$\int\limits_{\partial\Omega} \Big(u^h \cdot \sigma^{(n)}(u) - u \cdot \sigma^{(n)}(u^h)\Big)\, ds = \sum_{\pm} \int\limits_{\Lambda^h_\pm \setminus \Lambda_\pm} \Big(u \cdot \sigma^{(n)}(u^h) - u^h \cdot \sigma^{(n)}(u)\Big)\, ds$$

$$= \sum_{i=1}^{2} \int\limits_0^h [u_i^h](x_1)\sigma_{2i}(u; x_1, 0)\, dx_1. \tag{4.3}$$

Since $\sigma^{(n)}(u) = \sigma^{(n)}(u^h) = g$ on $\partial\Omega$, the left side I_l of formula (4.3) coincides with the increment in the work of the external forces and under Clapeyron's theorem, we have

$$I_l = \int\limits_{\partial\Omega} g \cdot (u^h - u)\, ds = -2\Delta U(h, 0). \tag{4.4}$$

As proved in [9] (see also [6]), near the tip $\mathcal{O}^h$ of an extended crack Λ^h, the field $u^h(x)$ is approximated with accuracy $O(h^{3/2})$ by the sum

$$c + K_1^\varepsilon U^{1\varepsilon}(x_1 - h, x_2) + K_2^\varepsilon U^{2\varepsilon}(x_1 - h, x_2), \tag{4.5}$$

where c is a constant vector; K_i^ε is the SIF for the initial position of the crack; and $(x_1 - h, x_2)$ are Cartesian coordinates with origin $\mathcal{O}^h$. To calculate the right side I_r of equality (4.3), we use the approximation (4.5) for u^h and the first representation (1.7) for u. Taking into account the normalization conditions (1.6) and (1.5) and the constraint (1.8) on the SIFs K_i^ε and K_i^σ we obtain

$$I_r = \frac{4}{\pi} B_{11,11} \sum_{i=1}^{2} K_i^\sigma K_i^\varepsilon \int\limits_0^h r^{-1/2}(h - r)^{1/2}\, dr$$

$$= 2hB_{11,11} \sum_{i=1}^{2} K_i^\sigma K_i^\varepsilon + o(h) = 2hB_{11,11} \sum_{i,j=1}^{2} T_{ij}^{-1} K_i^\varepsilon K_j^\varepsilon + o(h). \tag{4.6}$$

Here T_{ij}^{-1} are the elements of the matrix T^{-1} which is the reverse of T. Comparing expressions (4.4), (4.6), and (4.3) with the terms of the asymptotic formula (4.1), we arrive at the relation

$$M = 2B_{11,11}T^{-1}. \tag{4.7}$$

This, in particular, implies that the matrix T of conversion from the strain basis to the force basis is symmetric and positively defined.

The elements of the matrix $\mathcal{M}(\theta)$, which is also symmetric and positively defined [7], are the coefficients in the expansion at infinity

$$w^j(x) = U^j(x) + \sum_{k=1}^{2} \mathcal{M}_{jk}(\theta)V^k(x) + O(r^{-1}) \qquad (r \to \infty) \tag{4.8}$$

for the solution of the problem of a semi-infinite cut Λ with a branch $\Upsilon(\theta) = \{x\colon\ x_1 \in [0, \cos\theta],\ x_2 = x_1 \tan\theta\}$:

$$\begin{gathered} \partial_1\sigma_{1k}(w^j; x) - \partial_2\sigma_{2k}(w^j; x) = 0, \qquad x \in \mathbb{R}^2 \setminus (\Lambda \cup \Upsilon(\theta)), \quad k = 1, 2, \\ \sigma^{(n)}(w^j; x) = \sigma(w^j; x)n(x) = 0, \qquad x \in \Lambda^\pm \cup \Upsilon(\theta)^\pm. \end{gathered} \tag{4.9}$$

Here n is the unit outward normal vector,

$$n = (0, \mp 1) \quad \text{on} \quad \Lambda^\pm, \qquad n = (\pm\sin\theta, \mp\cos\theta)) \quad \text{on} \quad \Upsilon(\theta)^\pm. \tag{4.10}$$

We emphasize that according to relations (4.8) and (1.1), the displacements $w^j(x)$ increase at infinity, the solution of the homogeneous problem (4.9) is not trivial and the coefficients $\mathcal{M}_{jk}(\theta)$ for the damped components (1.10) are uniquely determined.

Let the loading be simple, i.e., the SIFs $K_j = \tau K_j^0$ be proportional to the total time-similar parameter $\tau > 0$. We assume that the crack is open and, in particular, $K_1^0 > 0$. According to the representation (4.1), the Griffith's energy criterion states that the crack branches in the direction θ_* at the moment τ_* if the following relations are satisfied:

$$
\begin{gathered}
F(\theta_*;\tau_*) = 0,\\
F(\theta;\tau) < 0 \quad \text{at any} \quad \tau < \tau_* \quad \text{and} \quad \theta.
\end{gathered} \tag{4.11}
$$

In this case,

$$
F(\tau;\theta) = \tau^2 \sum_{j,k=1}^{2} \mathcal{M}_{jk}(\theta) K_j K_k - 4\gamma(\theta), \tag{4.12}
$$

and $\gamma(\theta)$ is the surface energy density, which, generally speaking, depends on the direction of the branch $\Upsilon(\theta)$. Thus, the continuous function $F(\,\cdot\,;\theta_*)$ has a global maximum at the point $\theta = \theta_*$ (in the presence of a few such points, one should pose the question of crack bifurcation or branching). This interpretation of the energy fracture criterion is conventional (cf. [20, 21, 7] and other papers) and although for the case of complex loading and quasistatic development of a crack requires refinement, this interpretation is quite applicable to determine the deviation angle θ_*.

By virtue of the first condition in (4.11) and condition (4.2), straight crack growth ($\theta = 0$) corresponds to the following critical value of the loading parameter:

$$
\tau_0 = 2\gamma(0)^{1/2} \Big(\sum_{j,k=1}^{2} M_{jk} K_j K_k \Big)^{-1/2}. \tag{4.13}
$$

Formula (4.13) contains the SIFs and the coefficients M_{jk}, which are expressed in (1.15) in terms of the integrals (1.12) of the elements of the basis $\{U^{j\varepsilon}\}$. If

$$
\tau_0^2 \sum_{j,k=1}^{2} \mathcal{M}'_{jk}(0) K_j K_k \neq 4\gamma'(0), \tag{4.14}
$$

where the prime denotes the derivative with respect to the angle θ, the second condition in (4.11) is *a priori* violated, the crack necessarily deviates from the initial direction (its deviation is observed), fracture begins earlier, i.e., at $\tau_* < \tau_0$, and the critical load decreases.

Let us calculate the derivatives $\mathcal{M}'_{jk}(0)$ using the established properties of the strain basis. For this, we make the change of variables

$$
x \mapsto y = (y_1, y_2) = (x_1 - \cos\theta, x_2 - \sin\theta) \tag{4.15}
$$

and, considering the angle θ small, we extend the boundary conditions written in the second line of (4.9) to the half-line $L = \{y\colon\ y_1 \le 0,\ y_2 = 0\}$. For slightly curved, smooth and kinked cracks, a rectification method was used in [22–24, 12, 7] and other papers (in [25], an alternative approach was proposed); it was rigorously substantiated in [19, Chapter 5]. Confining ourselves to formal asymptotic constructions and referring to [19, 26] for their substantiation, we seek a solution of problem (4.9), (4.8) in the form

$$
w^j(x) = U^j(y) + \theta W^j(y) + \dots\ . \tag{4.16}
$$

Here and below, the dots designate elements that are of no significance for the calculations. Using McLaren's formula for the variable $\rho^{-1} := |y|^{-1}$ and taking into account relations (4.15) and (1.14), we bring the asymptotic condition (4.8) to the form

$$
U^j(y) + \theta W^j(y) + \dots = w^j(x) = U^j(x) + \sum_{k=1}^{2} \mathcal{M}_{jk}(\theta) V^k(x) + \dots
$$

$$= U^j(y) + \cos\theta\partial_1 U^j(y) + \sin\theta\partial_2 U^j(y) + \sum_{k=1}^{2} \mathcal{M}_{jk}(\theta)V^k(y) + \dots$$

$$= U^j(y) + \sum_{k=1}^{2}(\mathcal{M}_{jk}(0) - M_{jk})V^k(y) + \theta\Big\{\partial_2 U^j(y) + \sum_{k=1}^{2}\mathcal{M}'_{jk}(0)V^k(y)\Big\} + \dots .$$

From this we derive equality (4.2) and the relations

$$W^j(y) = \partial_2 U^j(y) + \sum_{k=1}^{2}\mathcal{M}'_{jk}(0)V^k(y) + O(\rho^{-1}), \qquad j = 1, 2. \tag{4.17}$$

Hence, the field $W^j(y)$ vanishes at infinity as $O(\rho^{-1/2})$. Of course, it satisfies the homogeneous equilibrium equations in (1.4). Seeking the boundary conditions on the crack faces L, according to the boundary condition written in the second line (4.9), for $k = 1, 2$ and $y_1 \in (-\infty, -1)$, we have

$$0 = \sigma_{2k}(w^j; x_1, \pm 0) = \sigma_{2k}(U^j + \theta W^j; y)\Big|_{x_2=\pm 0} + \dots$$

$$= \sigma_{2k}(U^j; y_1, -\sin\theta \pm 0) + \theta\sigma_{2k}(W^j; y_1, -\sin\theta \pm 0) + \dots$$

$$= \sigma_{2k}(U^j; y_1, \pm 0) + \theta\{\sigma_{2k}(W^j; y_1, \pm 0) - \partial_2\sigma_{2k}(U^j; y_1, \pm 0)\} + \dots .$$

Taking into account the boundary conditions in the problem (1.4) and the properties (2.2) and (2.3) of the vectors U^j, we find that

$$\begin{gathered}\sigma_{2k}(W^1; y_1, \pm 0) = 0, \quad k = 1, 2; \qquad \sigma_{22}(W^2; y_1, \pm 0) = 0,\\ \sigma_{21}(W^2; y_1, \pm 0) = -\partial_1\sigma_{11}(U^2; y_1, \pm 0), \qquad y_1 \in (-\infty, -1).\end{gathered} \tag{4.18}$$

By virtue of formula (4.10) for the normal n, similar operations with the stresses on the faces of the branch

$$0 = \sigma_{2k}(w^j; x_1, x_1\tan\theta \pm 0) = \sigma_{2k}(U^j + \theta W^j; y)\Big|_{x_2 = x_1\tan\theta \pm 0} + \dots$$

$$= \sigma_{2k}(U^j; y_1, \pm 0) + \theta\{\sigma_{2k}(W^j; y_1, \pm 0) - y_1\partial_2\sigma_{2k}(U^j; y_1, \pm 0)\} + \dots$$

give the boundary conditions

$$\begin{gathered}\sigma_{2k}(W^1; y_1, \pm 0) = 0, \quad k = 1, 2; \qquad \sigma_{22}(W^2; y_1, \pm 0) = 0,\\ \sigma_{21}(W^2; y_1, \pm 0) = y_1\partial_1\sigma_{11}(U^2; y_1, \pm 0) + \sigma_{11}(U^2; y_1, \pm 0), \qquad y_1 \in (-1, 0).\end{gathered} \tag{4.19}$$

As a result, we find that W^1 is a bounded solution of the problem (1.4) that vanishes at infinity, i.e., $W^1 = 0$, and the sum of the asymptotic terms indicated on the right of (4.17) is equal to zero in the case $j = 1$. Now, taking into account relations (2.6) and (1.14), where $j = 2$, we conclude that

$$\mathcal{M}'_{11}(0) = -M_{21} = -Q(\partial_1 U^1, U^2; \Gamma), \qquad \mathcal{M}'_{12}(0) = -M_{22} = -Q(\partial_1 U^2, U^2; \Gamma). \tag{4.20}$$

Since $\mathcal{M}(\theta)$ is a symmetric matrix, it remains to calculate the derivative $\mathcal{M}'_{22}(0)$. We use Green's formula for the solutions W^2 and U^2 in the circle $\{y : \rho = R\}$ with a radial cut; after simplifications, we obtain the equality

$$Q(W^2, U^2; \Gamma_R) = \sum_{\pm} \pm \int_{-R}^{0} U_1^2(y_1, \pm 0)\sigma_{21}(W^2; y_1, \pm 0)\, dy_1.$$

By virtue of formulas (2.1), (2.4) and (4.18), (4.19), the integrand on the upper and lower faces $L^\pm$ coincide and, consequently, $Q(W^2, U^2; \Gamma_R) = 0$. In the limit $R \to +\infty$, W^2 is replaced by the sum of the asymptotic terms indicated on the right side of formula (4.17), where $j = 2$, and as a result, using the normalization (1.11), we arrive at the relation

$$\mathcal{M}'_{22}(0) = Q(\partial_2 U^2, U^2; \Gamma_R). \tag{4.21}$$

If $\{x : x_2 = 0\}$ is a plane of elastic symmetry, expression (4.21) vanishes. We note that for the isotropic case,

$$M_{11} = M_{22} = (\lambda + 2\mu)[2\mu(\lambda + \mu)]^{-1}, \qquad M_{12} = M_{21} = 0,$$

where $\lambda \geq 0$, $\mu > 0$ are Lamé constants.

Relations (4.20) and (4.21) are similar in form to (1.15) but there is a large difference but between formulas (1.15), (4.20), and (4.21): in the first group of equalities, Γ is an simple arc that connects the crack faces and encompasses the tip, but on the right side of (4.21), the arc Γ_R should begin and terminate at the same point.

Thus, for arbitrary anisotropy, the deformation basis of the singular solutions (1.1) gives simple integral representations (4.2), (4.20) and (1.15), (4.21) for all elements of the matrices $\mathcal{M}(0)$ and $\mathcal{M}'(0)$ that appear in formulas (4.1), (4.13), and (4.14) and refer to the attributes of Griffith's criterion.

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REFERENCES

1. M. L. Williams, "Stress singularities resulting from various boundary conditions in angular corners of plates in extensions," *J. Appl. Mech.*, **19**, 526–528 (1952).
2. V. Z. Parton and P. I. Perlin, *Methods of the Mathematical Theory of Elasticity* [in Russian], Nauka, Moscow (1981).
3. A. A. Kulikov and S. A. Nazarov, "Conformity principle in plane problems of straight crack growth," *Izv. Ross. Akad. Nauk., Mekh. Tverd. Tela*, **1**, 77–87 (2004).
4. R. Duduchava and W. L. Wendland, "The Wiener–Hopf method for systems of pseudodifferential equations with an application to crack problems," *Integral Equations Operator Theory*, **23**, No. 3, 294–335 (1995).
5. M. Costabel and M. Dauge, "Crack singularities for general elliptic systems," *Math. Nachr.*, **235**, 29–49 (2002).
6. S. A. Nazarov, "Crack at an interface between anisotropic solids. Stress singularities and invariant integrals," *Prikl. Mat. Mekh.*, **62**, No. 3. 489–502 (1998).
7. I. I. Argatov and S. A. Nazarov, "Energy liberation due to crack kinking in a plane anisotropic solid," *Prikl. Mat. Mekh.*, **66**, No. 3, 502–514 (2002).
8. V. G. Maz'ya and B. A. Plamenevskii, "Coefficients in the asymptotic solutions of elliptic boundary-value problems in a region with conical points," *Math. Nachr.*, **76**, 29–60 (1977).
9. S. A. Nazarov, "Weight functions and invariant Integrals," in: *Computational Mechanics of a Deformable Solid* (collected scientific papers) [in Russian], No. 1 (1990), pp. 17–31.
10. S. A. Nazarov and O. R. Polyakova, "Weight functions and invariant integrals of higher orders," *Izv. Ross. Akad. Nauk., Mekh. Tverd. Tela*, **1**, 104–119 (1995).
11. S. A. Nazarov, "Possibility of calculating all factors of lower-order stress singularities using invariant integrals," *Vest. St. Petersburg. Gos. Univ.*, Ser. 1, No. 4(22) (1996), pp. 95–99.
12. A. B. Movchan, S. A. Nazarov, and O. R. Polyakova, "Curving of the trajectory for quasistatic crack growth in a plane with a small defect, " in: *Elasticity and Plasticity Research* (collected scientific papers) [in Russian], No. 18, St. Petersburg Gos. Univ., St. Petersburg (1999), pp. 142–161.
13. M. Comninou, "The interface crack," *Trans. ASME, Ser. E, J. Appl. Mech.*, **44**, No. 4, 631–636 (1977).
14. M. Comninou and D. Schmueser, "The interface crack in a combined tension-compression and shear field," *Trans. ASME, Ser. E, J. Appl. Mech.*, **46**, No. 2, 345–348 (1979).
15. J. Sokolowski and A. M. Khludnev, "On the derivative of the energy functional along the crack length in elasticity problems," *Prikl. Mat. Mekh.*, **64**, No. 3, 464–475 (2000).
16. A. M. Khludnev and J. Sokolowski, "Griffith formulae for elasticity systems with unilateral conditions in domains with cracks," *Europ. J. Mech. Solids*, **19**, No. 1, 105–120 (2000).
17. N. B. Alfutova, A. B. Movchan, and S. A. Nazarov, "Algebraic equivalence of plane problems for orthotropic and anisotropic media," *Vest. Leningrad. Univ.*, Ser. 1. No. 3(15), 64–68 (1991).
18. V. G. Maz'ya and S. A. Nazarov, "Asymptotic behavior of the energy integrals for small perturbations of the boundary near corner and conical points," *Tr. Mosk. Mat. Obshch.*, **50**, 79–129 (1987).
19. W. G. Mazja, S. A. Nasarow, B. A. Plamenewski, *Asymptotic Theory of Elliptic Boundary-Value Problems in Singularly Perturbed Domains*, Vol. 1, Basel Birkhäuser Verlag, Berlin (2000).
20. V. Z. Parton amd E. M. Morozov, *Mechanics of Elastoplastic Fracture* [in Russian], Nauka, Moscow (1974).

21. D. Leguillon and E. Sanchez-Palencia, “Fracture in heterogeneous materials. Weak and strong singularities,” in: *New Advances in Computational Structural Mechanics*, Elsevier Sci. Publications B. V., Paris (1992), pp. 423–434.
22. N. V. Bannichuk, “Determining the shape of a curved crack using the small-parameter method,” *Izv. Ross. Akad. Nauk., Mekh. Tverd. Tela*, **2**, 130–137 (1970).
23. B. Cotterel and J. R. Rice, “Slightly curved or kinked cracks,” *Int. J. Fract.*, **16**, No. 2, 155–169 (1980).
24. M. Amestoy and J. B. Leblond, “Crack path in plane situations. II. Detailed form of the expansion of the stress intensity factors,” *Int. J. Solids Struct.*, **29**, No. 4, 465–501 (1992).
25. R. V. Gol’dstein and R. L. Salganik, “Plane problem of curved cracks in solids, ” *Izv. Ross. Akad. Nauk., Mekh. Tverd. Tela*, **3**, 69–82 (1970).
26. C. Eck, S. A. Nazarov, and W. L. Wendland, “Asymptotic analysis for a mixed boundary-value contact problem,” *Arch. Rational Mech. Anal.*, **156**, 275–316 (2001).